GAUSS'S ABSTRACT OF THE DISQUISITIONES GENERALES CIRCA SUPERFICIES CURVAS, PRESENTED TO THE ROYAL SOCIETY OF GÖTTINGEN.

Göttingische gelehrte Anzeigen. No. 177. Pages 1761–1768. 1827. November 5.

On the 8th of October, Hofrath Gauss presented to the Royal Society a paper:

Disquisitiones generales circa superficies curvas.

Although geometers have given much attention to general investigations of curved surfaces and their results cover a significant portion of the domain of higher geometry, this subject is still so far from being exhausted, that it can well be said that, up to this time, but a small portion of an exceedingly fruitful field has been cultivated. Through the solution of the problem, to find all representations of a given surface upon another in which the smallest elements remain unchanged, the author sought some years ago to give a new phase to this study. The purpose of the present discussion is further to open up other new points of view and to develop some of the new truths which thus become accessible. We shall here give an account of those things which can be made intelligible in a few words. But we wish to remark at the outset that the new theorems as well as the presentations of new ideas, if the greatest generality is to be attained, are still partly in need of some limitations or closer determinations, which must be omitted here.

In researches in which an infinity of directions of straight lines in space is concerned, it is advantageous to represent these directions by means of those points upon a fixed sphere, which are the end points of the radii drawn parallel to the lines. The centre and the radius of this *auxiliary sphere* are here quite arbitrary. The radius may be taken equal to unity. This procedure agrees fundamentally with that which is constantly employed in astronomy, where all directions are referred to a fictitious celestial sphere of infinite radius. Spherical trigonometry and certain other theorems, to which the author has added a new one of frequent application, then serve for the solution of the problems which the comparison of the various directions involved can present.

[46] If we represent the direction of the normal at each point of the curved surface by the corresponding point of the sphere, determined as above indicated, namely, in this way, to every point on the surface, let a point on the sphere correspond; then, generally speaking, to every line on the curved surface will correspond a line on the sphere, and to every part of the former surface will correspond a part of the latter. The less this part differs from a plane, the smaller will be the corresponding part on the sphere. It is, therefore, a very natural idea to use as the measure of the total curvature, which is to be assigned to a part of the curved surface, the area of the corresponding part of the sphere. For this reason the author calls this area the *integral curvature* of the corresponding part of the curved surface. Besides the magnitude of the part, there is also at the same time its *position* to be considered. And this position may be in the two parts similar or inverse, quite independently of the relation of their magnitudes. The two cases can be distinguished by the positive or negative sign of the total curvature. This distinction has, however, a definite meaning only when the figures are regarded as upon definite sides of the two surfaces. The author regards the figure in the case of the sphere on the outside, and in the case of the curved surface on that side upon which we consider the normals erected. It follows then that the positive sign is taken in the case of convexo-convex or concavo-concave surfaces (which are not essentially different), and the negative in the case of concavo-convex surfaces. If the part of the curved surface in question consists of parts of these different sorts, still closer definition is necessary, which must be omitted here.

The comparison of the areas of two corresponding parts of the curved surface and of the sphere leads now (in the same manner as, *e. g.*, from the comparison of volume and mass springs the idea of density) to a new idea. The author designates as *measure of curvature* at a point of the curved surface the value of the fraction whose denominator is the area of the infinitely small part of the curved surface at this point and whose numerator is the area of the corresponding part of the surface of the auxiliary sphere, or the integral curvature of that element. It is clear that, according to the idea of the author, integral curvature and measure of curvature in the case of curved surfaces are analogous to what, in the case of curved lines, are called respectively amplitude and curvature simply. He hesitates to apply to curved surfaces the latter expressions, which have been accepted more from custom than on account of fitness. Moreover, less depends upon the choice of words than upon this, that their introduction shall be justified by pregnant theorems.

The solution of the problem, to find the measure of curvature at any point of a curved surface, appears in different forms according to the manner in which the nature of the curved surface is given. When the points in space, in general, are
[47] distinguished by three rectangular coordinates, the simplest method is to express one coordinate as a function of the other two. In this way we obtain the simplest expression for the measure of curvature. But, at the same time, there arises a remarkable relation between this measure of curvature and the curvatures of the curves formed by the intersections of the curved surface with planes normal to it. EULER, as is well known, first showed that two of these cutting planes which

intersect each other at right angles have this property, that in one is found the greatest and in the other the smallest radius of curvature; or, more correctly, that in them the two extreme curvatures are found. It will follow then from the above mentioned expression for the measure of curvature that this will be equal to a fraction whose numerator is unity and whose denominator is the product of the extreme radii of curvature. The expression for the measure of curvature will be less simple, if the nature of the curved surface is determined by an equation in x, y, z. And it will become still more complex, if the nature of the curved surface is given so that x, y, z are expressed in the form of functions of two new variables p, q. In this last case the expression involves fifteen elements, namely, the partial differential coefficients of the first and second orders of x, y, z with respect to p and q. But it is less important in itself than for the reason that it facilitates the transition to another expression, which must be classed with the most remarkable theorems of this study. If the nature of the curved surface be expressed by this method, the general expression for any linear element upon it, or for $\sqrt{dx^2 + dy^2 + dz^2}$, has the form $\sqrt{E\,dp^2 + 2F\,dp \cdot dq + G\,dq^2}$, where E, F, G are again functions of p and q. The new expression for the measure of curvature mentioned above contains merely these magnitudes and their partial differential coefficients of the first and second order. Therefore we notice that, in order to determine the measure of curvature, it is necessary to know only the general expression for a linear element; the expressions for the coordinates x, y, z are not required. A direct result from this is the remarkable theorem: If a curved surface, or a part of it, can be developed upon another surface, the measure of curvature at every point remains unchanged after the development. In particular, it follows from this further: Upon a curved surface that can be developed upon a plane, the measure of curvature is everywhere equal to zero. From this we derive at once the characteristic equation of surfaces developable upon a plane, namely,

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \cdot \partial y}\right)^2 = 0,$$

when z is regarded as a function of x and y. This equation has been known for some time, but according to the author's judgment it has not been established previously with the necessary rigor.

These theorems lead to the consideration of the theory of curved surfaces from [48] a new point of view, where a wide and still wholly uncultivated field is open to investigation. If we consider surfaces not as boundaries of bodies, but as bodies of which one dimension vanishes, and if at the same time we conceive them as flexible but not extensible, we see that two essentially different relations must be distinguished, namely, on the one hand, those that presuppose a definite form of the surface in space; on the other hand, those that are independent of the various forms which the surface may assume. This discussion is concerned with the latter. In accordance with what has been said, the measure of curvature belongs to this case. But it is easily seen that the consideration of figures constructed upon the surface, their angles, their areas and their integral curvatures, the joining of the points by means of shortest lines, and the like, also belong to this case.

All such investigations must start from this, that the very nature of the curved surface is given by means of the expression of any linear element in the form $\sqrt{E\,dp^2 + 2F\,dp \cdot dq + G\,dq^2}$. The author has embodied in the present treatise a portion of his investigations in this field, made several years ago, while he limits himself to such as are not too remote for an introduction, and may, to some extent, be generally helpful in many further investigations. In our abstract, we must limit ourselves still more, and be content with citing only a few of them as types. The following theorems may serve for this purpose.

If upon a curved surface a system of infinitely many shortest lines of equal lengths be drawn from one initial point, then will the line going through the end points of these shortest lines cut each of them at right angles. If at every point of an arbitrary line on a curved surface shortest lines of equal lengths be drawn at right angles to this line, then will all these shortest lines be perpendicular also to the line which joins their other end points. Both these theorems, of which the latter can be regarded as a generalization of the former, will be demonstrated both analytically and by simple geometrical considerations. *The excess of the sum of the angles of a triangle formed by shortest lines over two right angles is equal to the total curvature of the triangle.* It will be assumed here that that angle (57° 17′ 45″) to which an arc equal to the radius of the sphere corresponds will be taken as the unit for the angles, and that for the unit of total curvature will be taken a part of the spherical surface, the area of which is a square whose side is equal to the radius of the sphere. Evidently we can express this important theorem thus also: the excess over two right angles of the angles of a triangle formed by shortest lines is to eight right angles as the part of the surface of the auxiliary sphere, which corresponds to it as its integral curvature, is to the whole surface of the sphere. In general, the excess over $2n-4$ right angles of the angles of a polygon of n sides, if these are shortest lines, will be equal to the integral curvature of the polygon.

[49] The general investigations developed in this treatise will, in the conclusion, be applied to the theory of triangles of shortest lines, of which we shall introduce only a couple of important theorems. If a, b, c be the sides of such a triangle (they will be regarded as magnitudes of the first order); A, B, C the angles opposite; α, β, γ the measures of curvature at the angular points; σ the area of the triangle, then, to magnitudes of the fourth order, $\frac{1}{3}(\alpha+\beta+\gamma)\sigma$ is the excess of the sum $A+B+C$ over two right angles. Further, with the same degree of exactness, the angles of a plane rectilinear triangle whose sides are a, b, c, are respectively

$$A - \frac{1}{12}(2\alpha + \beta + \gamma)\sigma,$$
$$B - \frac{1}{12}(\alpha + 2\beta + \gamma)\sigma,$$
$$C - \frac{1}{12}(\alpha + \beta + 2\gamma)\sigma.$$

We see immediately that this last theorem is a generalization of the familiar theorem first established by LEGENDRE. By means of this theorem we obtain the angles of a plane triangle, correct to magnitudes of the fourth order, if we diminish each angle of the corresponding spherical triangle by one-third of the spherical excess. In the case of non-spherical surfaces, we must apply unequal reductions to the angles, and this inequality, generally speaking, is a magnitude of the third order. However, even if the whole surface differs only a little from the spherical form, it will still involve also a factor denoting the degree of the deviation from the spherical form. It is unquestionably important for the higher geodesy that we be able to calculate the inequalities of those reductions and thereby obtain the thorough conviction that, for all measurable triangles on the surface of the earth, they are to be regarded as quite insensible. So it is, for example, in the case of the greatest triangle of the triangulation carried out by the author. The greatest side of this triangle is almost fifteen geographical[1] miles, and the excess of the sum of its three angles over two right angles amounts almost to fifteen seconds. The three reductions of the angles of the plane triangle are $4''.95113$, $4''.95104$, $4''.95131$. Besides, the author also developed the missing terms of the fourth order in the above expressions. Those for the sphere possess a very simple form. However, in the case of measurable triangles upon the earth's surface, they are quite insensible. And in the example here introduced they would have diminished the first reduction by only two units in the fifth decimal place and increased the third by the same amount.

[1]This German geographical mile is four minutes of arc at the equator, namely, 7.42 kilometers, and is equal to about 4.6 English statute miles. [Translators.]